



# Self-duality in the class of precompact groups<sup>☆</sup>

Mikhail Tkachenko<sup>\*</sup>

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, México, D.F., Mexico  
 Departament de Matemàtiques, Universitat Jaume I, Campus del Riu Sec, Castelló de la Plana, E-12071, Spain

## ARTICLE INFO

### Article history:

Dedicated to Robert Lowen on the occasion of his 60th anniversary

### MSC:

primary 43A40, 22A05, 54H11  
 secondary 54D30, 54G20

### Keywords:

Reflexive  
 Dual group  
 Self-dual  
 MAP group  
 Precompact  
 Countably compact  
 Pseudocompact  
 Countably pseudocompact

## ABSTRACT

A topological Abelian group  $G$  is called (strongly) self-dual if there exists a topological isomorphism  $\Phi : G \rightarrow G^\wedge$  of  $G$  onto the dual group  $G^\wedge$  (such that  $\Phi(x)(y) = \Phi(y)(x)$  for all  $x, y \in G$ ). We prove that every countably compact self-dual Abelian group is finite. It turns out, however, that for every infinite cardinal  $\kappa$  with  $\kappa^\omega = \kappa$ , there exists a pseudocompact, non-compact, strongly self-dual Boolean group of cardinality  $\kappa$ .

© 2009 Elsevier B.V. All rights reserved.

## 1. Introduction

We work in a field that has its origin in the Pontryagin–van Kampen duality theory, but our main concern is outside of the class of locally compact groups. The first successful attempts to extend the Pontryagin–van Kampen duality to wider classes of topological Abelian groups were undertaken by Kaplan [15], Leptin [16], Venkataraman [20], and Banaszczyk [3], among others. This line of the study has been continued in different directions by Chasco and Martín-Peínador [5,6], Pestov [17,18], Hernández [11], Hernández and Uspenskij [13], Raczkowski and Trigos [19], Hernández and Macario [12], just to mention a few contributors to the area.

Each time a new class of reflexive groups was found, it turned out that the groups in that class were not precompact or, alternatively, were compact under the assumption of their precompactness. This fact led to the conjecture that every precompact reflexive Abelian group was compact.

It is shown in the recent article [1], however, that there exist pseudocompact non-compact reflexive topological groups and, consistently, even countably compact non-compact reflexive groups. Since, according to [7, Theorem 1.1], every pseu-

<sup>☆</sup> The author was supported by CONACyT of Mexico, grant 000000000074468.

<sup>\*</sup> Address for correspondence: Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, México, D.F., Mexico.

E-mail address: [mich@xanum.uam.mx](mailto:mich@xanum.uam.mx).

docompact topological group is precompact, the results from [1] solve the problem of the existence of precompact, non-compact reflexive groups in the affirmative. Here we complement the main results from [1] in two ways.

Let us call a topological Abelian group  $G$  *self-dual* if  $G$  is topologically isomorphic to the dual group  $G^\wedge$ . If a topological isomorphism  $\Phi : G \rightarrow G^\wedge$  can be chosen to satisfy the condition  $\Phi(x)(y) = \Phi(y)(x)$  for all  $x, y \in G$ , we call  $G$  *strongly self-dual*. The mapping  $\Phi$  is said to be *symmetric*. In the case of locally compact Abelian groups, symmetric isomorphisms appeared in [8].

Intuitively, every self-dual group  $G$  should be *reflexive*, i.e., the reflection mapping  $\alpha_G : G \rightarrow G^{\wedge\wedge}$  of  $G$  to the bidual group  $G^{\wedge\wedge}$  should be a topological isomorphism (see Section 2). This conjecture, however, is confirmed in Proposition 2.2 for strongly self-dual groups only.

Since the dual group  $G^\wedge$  of a compact group  $G$  is discrete, every self-dual compact group is finite (while all finite Abelian groups are self-dual). We apply two results from [12] to deduce in Corollary 2.6 that every countably compact self-dual group is finite, which generalizes the similar fact for compact groups. We also show in Proposition 2.7 that the weight and cardinality of every precompact self-dual group coincide. This implies that every infinite precompact self-dual group has uncountable weight and cardinality (see Corollary 2.8).

In Section 3 we construct, for every infinite cardinal  $\kappa$  satisfying  $\kappa^\omega = \kappa$ , a pseudocompact non-compact strongly self-dual Boolean group  $G$  of cardinality  $\kappa$ . Hence, in a sense, our series of examples of strongly self-dual pseudocompact groups in Theorem 3.3 is the best possible.

Theorem 2.2 of [1] states that if all compact subsets of a pseudocompact Abelian group are finite, then the group is reflexive. Our technique in the proof of Theorem 3.3 is quite different. It consists in constructing a symmetric  $\kappa \times \kappa$  matrix with entries in  $\mathbb{Z}(2) = \{1, -1\}$  that describes completely both the algebraic and topological structures of the groups  $G$  and  $G^\wedge \cong G$ , where  $\kappa = |G| = w(G)$ , the latter being the weight of  $G$ .

Since the completion of a precompact Boolean group is a compact Boolean group and completions preserve the weight of groups, we conclude that the completion of  $G$  is topologically isomorphic to the group  $\mathbb{Z}(2)^\kappa$  (see [14, Theorem 25.9] or [2, Theorem 9.6.26]). Hence the completion of  $G$  is zero-dimensional and so is the group  $G$ . It turns out, however, that there exist connected, pseudocompact strongly self-dual groups of arbitrarily large cardinality (see Remark 3.4). So, as in the case of Pontryagin–van Kampen's duality, the dimensional properties of pseudocompact groups are not determined by the strong self-duality of the groups.

### 1.1. Notation and terminology

The circle group  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  is the multiplicative subgroup of the complex plane  $\mathbb{C}$ . The group  $\mathbb{T}$  carries the topology inherited from  $\mathbb{C}$ . The group  $\mathbb{Z}(2) = \{1, -1\}$  is identified with the corresponding subgroup of  $\mathbb{T}$ .

An abstract group  $G$  with identity  $e$  is called *Boolean* if  $x^2 = e$ , for each  $x \in G$ . Clearly, the group  $\mathbb{Z}(2)$  and all powers of  $\mathbb{Z}(2)$  are Boolean groups.

Given a topological Abelian group  $G$ , we denote by  $G^\wedge$  the dual group of  $G$  consisting of all continuous homomorphisms of  $G$  to  $\mathbb{T}$ . The elements of  $G^\wedge$  are called *characters* of  $G$ . The dual group  $G^\wedge$  always carries the compact-open topology whose base at the neutral element of  $G^\wedge$  is formed by the sets

$$K^\triangleright = \{\chi \in G^\wedge : \chi(K) \subset \mathbb{T}_+\},$$

where  $K$  is a compact subset of  $G$  and  $\mathbb{T}_+ = \{z \in \mathbb{T} : \operatorname{Re}(z) > 0\}$ .

The Raïkov completion of a topological group  $G$  is denoted by  $\varrho G$ . It is well known that  $G$  is a dense topological subgroup of the complete group  $\varrho G$ , and that  $\varrho G$  is compact if and only if the group  $G$  is precompact (i.e.,  $G$  can be covered by finitely many translates of every neighbourhood of the identity in  $G$ ).

A subgroup  $H$  of a topological Abelian group  $G$  is called  *$h$ -embedded* in  $G$  provided every homomorphism  $h : H \rightarrow \mathbb{T}$  can be extended to a continuous homomorphism  $\tilde{h} : G \rightarrow \mathbb{T}$ . If the group  $G$  is precompact, a subgroup  $H$  of  $G$  is  *$h$ -embedded* if and only if  $H$  inherits from  $G$  the *biggest* precompact topological group topology.

The weight of a topological group  $G$  (i.e., the minimum cardinality of a base for the group) is denoted by  $w(G)$ . The power of the continuum is denoted by  $\mathfrak{c}$ , so  $\mathfrak{c} = 2^\omega$ .

## 2. Some facts about self-duality

First we will establish that every strongly self-dual group is reflexive. Let us recall that for a topological Abelian group  $G$ , the *reflection mapping*  $\alpha_G : G \rightarrow G^{\wedge\wedge}$  is defined by  $\alpha_G(x)(\chi) = \chi(x)$ , for all  $x \in G$  and  $\chi \in G^\wedge$ . The group  $G$  is *reflexive* if  $\alpha_G$  is a topological isomorphism of  $G$  onto  $G^{\wedge\wedge}$ .

A topological Abelian group  $G$  is called *maximally almost periodic* or, briefly, a *MAP* group if  $G^\wedge$  separates elements of  $G$ , i.e., for every  $x \in G$  distinct from the identity of  $G$ , there exists  $\chi \in G^\wedge$  such that  $\chi(x) \neq 1$ . This is equivalent to saying that the reflection mapping  $\alpha_G$  is injective. Every precompact Abelian group  $G$  is *MAP* since, by Peter–Weyl's theorem,  $G$  has sufficiently many characters (see [2, Theorem 9.4.11]). It is also clear that every reflexive group is *MAP*.

**Lemma 2.1.** *Every self-dual topological Abelian group is MAP.*

**Proof.** It is easy to see that the dual group  $G^\wedge$  is MAP, for any topological group  $G$ . Indeed, if a character  $\chi \in G^\wedge$  is distinct from the neutral element of  $G^\wedge$ , there exists  $x \in G$  such that  $\chi(x) \neq 1$ . Then  $\alpha_G(x) \in G^{\wedge\wedge}$  and  $\alpha_G(x)(\chi) = \chi(x) \neq 1$ . Therefore, the elements of  $G^{\wedge\wedge}$  separate points of  $G^\wedge$ . It remains to note that topological isomorphisms preserve the property of being MAP.  $\square$

We apply Lemma 2.1 for the proof of the following fact:

**Proposition 2.2.** *Let  $G$  be a strongly self-dual topological group. Then there exist symmetric topological isomorphisms  $\Phi : G \rightarrow G^\wedge$  and  $\Psi : G^\wedge \rightarrow G^{\wedge\wedge}$  such that  $\alpha_G = \Psi \circ \Phi$ . Hence every strongly self-dual topological group is reflexive.*

**Proof.** The group  $G$  being strongly self-dual, there exists a topological isomorphism  $\Phi : G \rightarrow G^\wedge$  such that  $\Phi(x)(y) = \Phi(y)(x)$  for all  $x, y \in G$ . Let  $\Phi^\wedge : G^{\wedge\wedge} \rightarrow G^\wedge$  be the dual homomorphism defined by  $\Phi^\wedge(\varphi) = \varphi \circ \Phi$ , for each  $\varphi \in G^{\wedge\wedge}$ . Since  $\Phi$  is a topological isomorphism, so is  $\Phi^\wedge$ .

We define a mapping  $\Psi : G^\wedge \rightarrow G^{\wedge\wedge}$  by

$$\Psi(\chi)(h) = h(\Phi^{-1}(\chi)),$$

for all  $\chi, h \in G^\wedge$ . Our definition is correct in the sense that  $\Psi(\chi)$  is a character of  $G^\wedge$ , for each  $\chi \in G^\wedge$ . It is also immediate from the definition that  $\Psi$  is a homomorphism.

**Claim 1.**  $\Psi(\chi)(h) = \chi(\Phi^{-1}(h)) = \Psi(h)(\chi)$ , for all  $\chi, h \in G^\wedge$ . Hence  $\Psi$  is symmetric and  $\Psi(\chi) = \chi \circ \Phi^{-1}$  for each  $\chi \in G^\wedge$ .

Indeed, given  $\chi, h \in G^\wedge$ , take  $x, y \in G$  such that  $\Phi(x) = \chi$  and  $\Phi(y) = h$ . Then

$$\Psi(\chi)(h) = h(\Phi^{-1}(\chi)) = \Phi(y)(x) = \Phi(x)(y) = \chi(\Phi^{-1}(h)) = \Psi(h)(\chi),$$

whence the claim follows.

**Claim 2.**  $\Phi^\wedge \circ \Psi$  is the identity isomorphism of  $G^\wedge$  onto itself.

Indeed, take any  $\chi \in G^\wedge$ . It follows from Claim 1 that

$$(\Phi^\wedge \circ \Psi)(\chi) = \Psi(\chi) \circ \Phi = \chi \circ \Phi^{-1} \circ \Phi = \chi.$$

This proves Claim 2.

We now have that  $\Phi^\wedge$  and  $\Phi^\wedge \circ \Psi = id_{G^\wedge}$  are topological isomorphisms, whence it follows that so is  $\Psi = (\Phi^\wedge)^{-1}$ .

It remains to show that  $\Psi \circ \Phi = \alpha_G$ . Take arbitrary elements  $x \in G$  and  $\chi \in G^\wedge$ . Then, by Claim 1,

$$\begin{aligned} (\Psi \circ \Phi)(x)(\chi) &= \Psi(\Phi(x))(\chi) = \Psi(\chi)(\Phi(x)) \\ &= (\chi \circ \Phi^{-1})(\Phi(x)) = \chi(x) = \alpha_G(x)(\chi). \end{aligned}$$

It follows that  $\Psi(\Phi(x)) = \alpha_G(x)$  for each  $x \in G$ . This implies the equality  $\Psi \circ \Phi = \alpha_G$ . Since both  $\Phi$  and  $\Psi$  are topological isomorphisms, so is  $\alpha_G$ . Hence the group  $G$  is reflexive.  $\square$

Our next step is to establish that self-dual countably compact groups are finite. In fact, we will prove a slightly more general result in Proposition 2.5 which requires a lemma below:

**Lemma 2.3.** *Let  $G$  be a self-dual pseudocompact group. Then every countable subgroup of  $G$  is  $h$ -embedded and all compact subsets of  $G$  are finite.*

**Proof.** The dual group  $G^\wedge$  is topologically isomorphic to  $G$  and, hence, is pseudocompact. Therefore, by [7, Theorem 1.1], the group  $G^\wedge$  is precompact. Again, since  $G$  is pseudocompact, [12, Proposition 3.4] implies that every countable subgroup  $H$  of  $G^\wedge$  inherits from  $G^\wedge$  a topology which is finer than the biggest precompact topological group topology of the (abstract) group  $H$ . Evidently, an arbitrary subgroup of a precompact topological group is precompact, so we conclude that every countable subgroup  $H$  of  $G^\wedge$  carries the biggest precompact topological group topology, i.e.,  $H$  is  $h$ -embedded in  $G^\wedge$ . Hence the same conclusion is valid for countable subgroups of  $G$ . It now follows from [1, Proposition 2.1] that all compact subsets of  $G$  are finite.  $\square$

**Remark 2.4.** Very recently, E. Martín-Peinador proved that for a MAP group  $G$ , the dual group  $G^\wedge$  is precompact if and only if all compact subsets of  $G$  are finite. In particular, every compact subset of a precompact self-dual group is finite. This fact can be used to give an alternative proof of Lemma 2.3.

Let us recall that a topological group  $G$  is *countably pseudocompact* if for every countable set  $A \subseteq G$ , there exists a countable set  $B \subseteq G$  such that  $A \subseteq \overline{B}$  and  $\overline{B}$  is pseudocompact (see [12, Definition 4.1]). It is clear that every countably compact group is countably pseudocompact and that countably pseudocompact groups are pseudocompact. None of the implications is invertible.

**Proposition 2.5.** *Every countably pseudocompact self-dual group  $G$  is finite.*

**Proof.** Since countable pseudocompactness implies pseudocompactness, the groups  $G$  and  $G^\wedge$  are pseudocompact. The group  $G$  being countably pseudocompact, it follows from [12, Corollary 4.3] that the dual group  $G^\wedge$  endowed with the topology  $\omega(G^\wedge, G)$  of pointwise convergence on elements of  $G$  is a  $\mu$ -space, i.e., the closure of every functionally bounded subset of the space  $(G^\wedge, \omega(G^\wedge, G))$  is compact. Since, by Lemma 2.3, every compact subset of  $G$  is finite, we conclude that the compact-open topology of  $G^\wedge$  and the pointwise convergence topology  $\omega(G^\wedge, G)$  coincide. Thus  $G^\wedge$  is a  $\mu$ -space. But pseudocompact  $\mu$ -spaces are compact, whence it follows that the dual group  $G^\wedge$  is compact. Hence  $G \cong G^\wedge$  is a compact self-dual group and, therefore,  $G$  is finite.  $\square$

Since every finite discrete Abelian group is self-dual, we can reformulate Proposition 2.5 by saying that a countably pseudocompact Abelian group is self-dual if and only if it is finite.

**Corollary 2.6.** *Every countably compact self-dual group is finite.*

Self-duality is a strong property, especially in precompact groups. It has to imply non-trivial relations between cardinal invariants of the groups in this class. It is not surprising, therefore, that the weight and the cardinality of self-dual precompact groups coincide:

**Proposition 2.7.** *The equality  $w(G) = |G|$  holds for every infinite precompact self-dual group  $G$ .*

**Proof.** Let  $G$  be an infinite precompact self-dual group. Then the Raïkov completion  $\varrho G$  of  $G$  is a compact group. We claim that  $w(\varrho G) = w(G)$ . Indeed, denote by  $\chi(G)$  the minimum cardinality of a base for  $G$  at the neutral element. Since the space  $\varrho G$  is regular, it follows from [10, 2.1.C(a)] that  $\chi(\varrho G) = \chi(G)$ . The precompact groups  $G$  and  $\varrho G$  satisfy  $w(G) = \chi(G)$  and  $w(\varrho G) = \chi(\varrho G)$  according to [2, Corollary 5.2.4]. Hence  $w(\varrho G) = w(G)$ .

Since  $G$  is dense in  $\varrho G$  and every character  $\chi \in G^\wedge$  extends to a character of  $\varrho G$ , we can identify the dual groups  $G^\wedge$  and  $(\varrho G)^\wedge$  algebraically. By [2, Theorem 9.6.6], the compact group  $\varrho G$  satisfies  $w(\varrho G) = |(\varrho G)^\wedge|$ . Since  $G$  is self-dual and  $w(\varrho G) = w(G)$ , we conclude that  $w(G) = |G^\wedge| = |G|$ .  $\square$

It turns out that infinite precompact self-dual groups are never metrizable:

**Corollary 2.8.** *Every infinite precompact self-dual group has uncountable weight and cardinality.*

**Proof.** Suppose to the contrary that  $G$  is an infinite precompact self-dual group of countable weight. Let  $K = \varrho G$  be the Raïkov completion of  $G$ . Then  $K$  is a compact metrizable group and, since  $G$  is dense in  $K$ , it follows from [4, Theorem 2] that the dual groups  $G^\wedge$  and  $K^\wedge$  are topologically isomorphic. Since  $K^\wedge$  is discrete, the precompact group  $G \cong G^\wedge$  is finite, which is a contradiction. Hence  $w(G) > \omega$ .

Applying Proposition 2.7, we conclude that  $|G| = w(G) > \omega$ .  $\square$

### 3. Pseudocompact strongly self-dual groups

We show in Theorem 3.3 below that, unlike in the case of countably compact groups, there exist many pseudocompact strongly self-dual groups. The idea of our construction in Theorem 3.3 is suggested by the following lemma. In the sequel  $\tau_d$  stands for the discrete topology on a corresponding group.

Notice that for a Boolean group  $C$ , the dual group  $C^\wedge$  is canonically identified with a subgroup of  $\mathbb{Z}(2)^C$  since  $\chi(x) \in \{1, -1\}$ , for all  $\chi \in C^\wedge$  and  $x \in C$ .

**Lemma 3.1.** *Let  $G$  be a subgroup of the product group  $\mathbb{Z}(2)^H$ , where  $H$  is a Boolean group used as the index set. Suppose that  $G$  has the property that  $\pi_C(G) = (C, \tau_d)^\wedge$ , for every countable subgroup  $C$  of  $H$ , where  $\pi_C : \mathbb{Z}(2)^H \rightarrow \mathbb{Z}(2)^C$  is the restriction mapping (projection). Then the group  $G$  is pseudocompact and, for every character  $\chi \in G^\wedge$ , there exists an element  $h \in H$  such that  $\chi(x) = x(h)$  for each  $x \in G$ .*

**Proof.** To deduce that  $G$  is pseudocompact it suffices, by [7, Theorem 1.2], to verify that  $G$  intersects every non-empty  $G_\delta$ -set in its completion  $\varrho G$  or, equivalently, in its closure  $\overline{G}$  in  $\mathbb{Z}(2)^H$ . Since every countable subset  $B$  of  $H$  is contained in

a countable subgroup  $C$  of  $H$  and the projection  $\pi_C(G) = (C, \tau_d)^\wedge$  is compact, we conclude that  $\pi_B(G)$  is compact as well. This implies immediately that  $G$  intersects every non-empty  $G_\delta$ -set in  $\bar{G}$  and, hence, is pseudocompact.

Consider an arbitrary character  $\chi \in G^\wedge$ . Then  $\chi$  is a continuous homomorphism of the subgroup  $G$  of  $\mathbb{Z}(2)^H$  to the discrete group  $\mathbb{Z}(2)$ , so we can apply [2, Lemma 8.5.4] to find a countable set  $D \subseteq H$  and a character  $\chi_0$  on  $\pi_D(G)$  such that  $\chi = \chi_0 \circ \pi_D \upharpoonright G$ , where  $\pi_D : \mathbb{Z}(2)^H \rightarrow \mathbb{Z}(2)^D$  is the projection. Replacing  $D$ , if necessary, by the subgroup  $\langle D \rangle$  of  $H$  generated by  $D$ , we can assume without loss of generality that  $D$  is a subgroup of  $H$ .

Then  $K = \pi_D(G) = (D, \tau_d)^\wedge$  is a compact group and  $\chi_0 \in K^\wedge$ . Therefore, by Pontryagin–van Kampen's theorem, there exists  $h \in D$  such that  $\chi_0(y) = y(h)$ , for each  $y \in K$ . It follows from the equality  $\chi = \chi_0 \circ \pi_D \upharpoonright G$  that  $\chi(x) = x(h)$ , for each  $x \in G$ .  $\square$

Reformulating Lemma 3.1, we can say that every character  $\chi$  on  $G$  naturally identifies with the projection of  $\mathbb{Z}(2)^H$  to the  $h$ th factor  $\mathbb{Z}(2)$ , for some  $h \in H$ . Equivalently,  $\chi$  identifies with the element  $h \in H$  itself.

Let an infinite cardinal  $\kappa$  satisfy  $\kappa^\omega = \kappa$ . Take any Boolean group  $H$  of cardinality  $\kappa$ . Then  $H$  is algebraically isomorphic to the group  $\mathbb{Z}(2)^{(\kappa)}$ , the direct sum of  $\kappa$  copies of the group  $\mathbb{Z}(2)$ . Denote by  $X$  an independent subset of  $H$  that generates the group  $H$ . In other words,  $X$  is an algebraic basis for  $H$ . It is clear that  $|X| = \kappa$ . We also put

$$\mathcal{Y} = \{Y \subseteq X : |Y| = \omega\}.$$

With this notation, we will prove the following lemma:

**Lemma 3.2.** *There exist enumerations  $X = \{x_\alpha : \alpha < \kappa\}$  and  $\mathcal{Y} = \{Y_\alpha : \alpha < \kappa\}$  satisfying the following conditions for all  $\alpha, \beta < \kappa$ :*

- (a)  $x_\alpha \neq x_\beta$  if  $\alpha \neq \beta$ ;
- (b) every  $Y \in \mathcal{Y}$  appears  $\mathfrak{c}$  times in the above enumeration of  $\mathcal{Y}$ ;
- (c)  $x_\alpha \notin Y_\alpha$ ;
- (d) if  $\alpha \neq \beta$ , then either  $x_\alpha \notin Y_\beta$  or  $x_\beta \notin Y_\alpha$ .

**Proof.** Let  $\{y_\alpha : \alpha < \kappa\}$  and  $\{Z_\alpha : \alpha < \kappa\}$  be faithful enumerations of  $X$  and  $\mathcal{Y}$ , respectively. Take a function  $f : \kappa \rightarrow \kappa$  such that  $|f^{-1}(\gamma)| = \mathfrak{c}$  for each  $\gamma \in \kappa$ . We are going to construct the required enumerations of  $X$  and  $\mathcal{Y}$  by recursion.

Let  $Y_0 = Z_{f(0)}$  and put  $x_0 = y_\gamma$ , where  $\gamma$  is the minimal ordinal less than  $\kappa$  such that  $y_\gamma \notin Y_0$ . Suppose that for some non-zero ordinal  $\alpha < \kappa$ , we have defined two families  $\{x_\mu : \mu < \alpha\}$  and  $\{Y_\mu : \mu < \alpha\}$  satisfying the following conditions for all  $\mu, \nu < \alpha$ :

- (i)  $x_\mu \neq x_\nu$  if  $\mu \neq \nu$ ;
- (ii)  $Y_\mu = Z_{f(\mu)}$ ;
- (iii)  $x_\mu \notin Y_\mu$ ;
- (iv) if  $\mu \neq \nu$ , then either  $x_\mu \notin Y_\nu$  or  $x_\nu \notin Y_\mu$ ;
- (v)  $x_\mu = y_\gamma$ , where  $\gamma < \kappa$  is the minimal possible ordinal for which conditions (i), (iii), and (iv) hold at the stage  $\mu$  of the construction, for such a choice of  $x_\mu$ .

Clearly,  $x_0$  and  $Y_0$  satisfy (i)–(v). It is easy to verify that, in the end of the construction, (i)–(iv) will imply conditions (a)–(d) of the lemma. Notice that (ii) is designed to guarantee (b). Having finished the construction, we will apply (v) to verify that  $X = \{x_\alpha : \alpha < \kappa\}$ .

To satisfy (ii) at the stage  $\alpha$ , we put  $Y_\alpha = Z_{f(\alpha)}$ . Denote by  $\gamma$  the minimal ordinal less than  $\kappa$  with the property that

$$y_\gamma \notin Y_\alpha \cup \bigcup \{Y_\mu : \mu < \alpha, x_\mu \in Y_\mu\} \cup \{x_\mu : \mu < \alpha\},$$

and put  $x_\alpha = y_\gamma$ . Since the sets  $Y_\mu$  with  $\mu \leq \alpha$  are countable and  $\alpha < \kappa$ , such an ordinal  $\gamma < \kappa$  does exist. It is easy to see that (i)–(v) hold at this step. This finishes our definition of the families  $\{x_\alpha : \alpha < \kappa\}$  and  $\{Y_\alpha : \alpha < \kappa\}$ .

Finally, we show that  $X = \{x_\alpha : \alpha < \kappa\}$ . Suppose not, and take the minimal ordinal  $\gamma_0 < \kappa$  such that  $y_{\gamma_0} \neq x_\alpha$  for each  $\alpha < \kappa$ . Let

$$T = \{\mu < \kappa : y_{\gamma_0} \notin Y_\mu\}.$$

Since  $Y_\mu$ 's run over all countable infinite subsets of  $X$ , we have that  $|T| = \kappa$ . Denote by  $A$  the set of all ordinals  $\alpha < \kappa$  such that

$$Y_\alpha \subseteq \{x_\mu : \mu \in T\}.$$

Then  $|A| = \kappa$ , so (i) implies that there exists  $\alpha_0 \in A$  such that  $x_{\alpha_0} = y_\gamma$  for some  $\gamma > \gamma_0$ . However, this contradicts condition (v) at the step  $\alpha_0$  of our construction, since the choice  $x_{\alpha_0} = y_{\gamma_0}$  was compatible with (i), (iii), and (iv) at that step. Indeed, it follows from our definition of  $\gamma_0$  that  $y_{\gamma_0} \neq x_\mu$ , for each  $\mu < \kappa$  (hence, for each  $\mu < \alpha_0$ ). Thus the alternative choice

$x_{\alpha_0} = y_{\gamma_0}$  was compatible with (i). It was compatible with (iii) as well, since the definition of  $\gamma_0$  and  $A$  together with  $\alpha_0 \in A$  imply that  $y_{\gamma_0} \notin Y_{\alpha_0}$ . The compatibility with (iv) is not difficult either. Indeed, take any  $\mu < \alpha_0$  such that  $x_\mu \in Y_{\alpha_0}$ . Since  $\alpha_0 \in A$ , we have that  $\mu \in T$ . Then the definition of  $T$  implies that  $y_{\gamma_0} \notin Y_\mu$ , as required. This finishes the proof of the lemma.  $\square$

Note that by Corollary 2.6, the group  $G$  in the following theorem cannot be either compact or countably compact.

**Theorem 3.3.** *Let  $\kappa$  be an infinite cardinal with  $\kappa^\omega = \kappa$ . There exists a pseudocompact strongly self-dual Boolean group  $G$  satisfying  $|G| = w(G) = \kappa$ .*

**Proof.** Let  $X$  be an algebraic basis of a Boolean group  $H$  satisfying  $|X| = |H| = \kappa$ . According to Lemma 3.2 (especially, taking into account item (b) of the lemma), there exist enumerations  $X = \{x_\alpha : \alpha < \kappa\}$  and  $\{(Y_\alpha, h_\alpha) : \alpha < \kappa\}$  of all pairs  $(Y, h)$ , where  $Y$  is a countable infinite subset of  $X$  and  $h$  is a mapping of  $Y$  to  $\mathbb{Z}(2)$ , such that  $X = \{x_\alpha : \alpha < \kappa\}$  and  $\mathcal{Y} = \{Y_\alpha : \alpha < \kappa\}$  satisfy conditions (a)–(d) of the lemma. Notice that we do not impose restrictions on the mappings  $h_\alpha$  at all.

With these enumerations in mind, we are going to construct a matrix  $N = (i_{\alpha, \beta})_{\alpha, \beta < \kappa}$  with entries  $i_{\alpha, \beta} \in \mathbb{Z}(2)$ . The matrix  $N$  will satisfy the following conditions for all  $\alpha, \beta < \kappa$ :

- (A)  $i_{\alpha, \beta} = i_{\beta, \alpha}$ ;
- (B) if  $x_\beta \in Y_\alpha$ , then  $i_{\alpha, \beta} = h_\alpha(x_\beta)$ .

Condition (A) implies that the matrix  $N$  is symmetric, while (B) prepares an application of Lemma 3.1.

Let us put  $R_0 = \emptyset$ . Suppose that for some non-zero  $\alpha < \kappa$  we have defined sets  $R_\beta \subseteq \kappa$  with  $\beta < \alpha$  and entries  $\{i_{\mu, \nu} : \mu, \nu \in R_\beta\}$  of  $N$  satisfying the following conditions for all  $\beta, \gamma < \alpha$ :

- (1)  $|R_\beta| \leq |\beta| \cdot \omega$ ;
- (2)  $R_\gamma \subseteq R_\beta$  if  $\gamma < \beta$ ;
- (3) if  $\mu \in R_\beta$  and  $x_\nu \in Y_\mu$ , then  $\nu \in R_\beta$ ;
- (4)  $\beta \subseteq R_\beta$ ;
- (5)  $i_{\mu, \nu} = i_{\nu, \mu}$  for all  $\mu, \nu \in R_\beta$ ;
- (6)  $i_{\mu, \nu} = h_\mu(x_\nu)$  whenever  $\mu \in R_\beta$  and  $x_\nu \in Y_\mu$ .

Note that conditions (1)–(6) hold vacuously true at the step 0. Clearly, (5) and (6) correspond to the above conditions (A) and (B), respectively. Conditions (1)–(4) describe properties of the sets  $R_\beta$ .

We put  $R_\alpha^* = \bigcup_{\beta < \alpha} R_\beta$ . Then, according to (1), we have that  $|R_\alpha^*| \leq |\alpha| \cdot \omega$ . The entries  $i_{\mu, \nu}$  of the matrix  $N$  have been defined for all  $\mu, \nu \in R_\alpha^*$  at the previous steps.

Put  $R'_\alpha = R_\alpha^* \cup \alpha$ . Since the sets  $Y_\mu$  are countable, we can apply the usual saturation process to find a subset  $R_\alpha$  of  $\kappa$  with  $|R_\alpha| \leq |\alpha| \cdot \omega$  such that  $R'_\alpha \subseteq R_\alpha$  and  $\nu \in R_\alpha$  holds whenever  $\mu \in R_\alpha$  and  $x_\nu \in Y_\mu$ . All this corresponds to conditions (1)–(4) of our recursive construction.

Let us define entries  $i_{\mu, \nu}$  for the pairs  $(\mu, \nu) \in (R_\alpha \times R_\alpha) \setminus (R_\alpha^* \times R_\alpha^*)$ . Conditions (5) and (6) force us to put  $i_{\nu, \mu} = i_{\mu, \nu} = h_\mu(x_\nu)$  whenever  $x_\nu \in Y_\mu$ . Let us verify that our definitions are consistent. Indeed, we would have a trouble if  $i_{\mu, \nu} = h_\mu(x_\nu) \neq h_\nu(x_\mu) = i_{\nu, \mu}$ . But this would imply that  $x_\nu \in Y_\mu$  and  $x_\mu \in Y_\nu$ , which is impossible by (d) of Lemma 3.2.

Clearly, we obtain the entries  $i_{\mu, \nu}$  on a symmetric part of  $N$ , i.e.,  $i_{\mu, \nu}$  is defined iff  $i_{\nu, \mu}$  is defined. Therefore, we complete the definitions by letting  $i_{\mu, \nu} = i_{\nu, \mu} = 1$  for the rest of pairs  $(\mu, \nu) \in R_\alpha \times R_\alpha$ . This finishes our construction of the matrix  $N$ .

Let  $\Psi_N : X \times X \rightarrow \mathbb{Z}(2)$  be a mapping defined by  $\Psi_N(x_\alpha, x_\beta) = i_{\alpha, \beta}$  for all  $\alpha, \beta < \kappa$ . It follows from condition (5) of the construction that  $\Psi_N(x, y) = \Psi_N(y, x)$  for all  $x, y \in X$ . Since  $X$  is an algebraic basis for  $H$ ,  $\Psi_N$  extends to a mapping  $\Psi : H \times H \rightarrow \mathbb{Z}(2)$  such that  $\Psi(g, h_1 \cdot h_2) = \Psi(g, h_1) \cdot \Psi(g, h_2)$  for all  $g, h_1, h_2 \in H$ . Then the symmetry of  $\Psi_N$  implies that  $\Psi$  is also symmetric, so that  $\Psi(g_1 \cdot g_2, h) = \Psi(g_1, h) \cdot \Psi(g_2, h)$  for all  $g_1, g_2, h \in H$ .

We thus have a symmetric matrix  $M = (i_{g, h})_{g, h \in H}$  with the entries  $i_{g, h} = \Psi(g, h) \in \mathbb{Z}(2)$ . It follows from the definition of  $\Psi$  that for every  $g \in H$ , the mapping  $\varphi_g : H \rightarrow \mathbb{Z}(2)$  defined by  $\varphi_g(h) = \Psi(g, h)$  for each  $h \in H$ , is a homomorphism. In other words,  $\varphi_g \in H^\wedge$ . Here  $H^\wedge$  corresponds to the group  $H$  endowed with the discrete topology.

**Claim 1.** *For every countable subgroup  $C$  of  $H$  and every homomorphism  $f : C \rightarrow \mathbb{Z}(2)$ , there exists  $x \in X$  such that  $\varphi_x|_C = f$ .*

Indeed, since  $C$  is countable, we can find a countable infinite set  $Y \subseteq X$  such that  $C \subseteq \langle Y \rangle$ . Since the group  $H$  is Boolean,  $f$  admits an extension to a homomorphism  $\tilde{f} : \langle Y \rangle \rightarrow \mathbb{Z}(2)$ . Denote by  $h$  the restriction of  $\tilde{f}$  to  $Y$ . Then  $(Y, h) = (Y_\alpha, h_\alpha)$  for some  $\alpha < \kappa$ . Let us verify that  $\varphi_{x_\alpha}$  and  $f$  coincide on  $C$ . Since  $\tilde{f}$  extends  $f$  and  $h = \tilde{f}|_Y$ , it suffices to show that  $h = \varphi_{x_\alpha}|_Y$ . By condition (6) of our recursive construction, we have for every  $x_\nu \in Y_\alpha$  that

$$\varphi_{x_\alpha}(x_\nu) = \Psi(x_\alpha, x_\nu) = i_{\alpha, \nu} = h_\alpha(x_\nu) = h(x_\nu).$$

Therefore,  $\varphi_{x_\alpha}$  extends  $h$  and coincides with  $f$  on  $C$ . This proves Claim 1.

Consider the mapping  $\Phi : H \rightarrow H^\wedge$  defined by  $\Phi(g) = \varphi_g$  for each  $g \in H$ . If  $x_1, \dots, x_n$  are pairwise distinct elements of  $X$  and  $g = x_1 \cdots x_n$ , the definition of  $M$  implies that  $\varphi_g(h) = \varphi_{x_1}(h) \cdots \varphi_{x_n}(h)$  for each  $h \in H$ , i.e.,  $\varphi_g = \varphi_{x_1} \cdots \varphi_{x_n}$ . This shows that  $\Phi$  is a homomorphism.

It also follows from the definition that  $\Phi$  is symmetric. Indeed, take arbitrary elements  $x, y \in H$ . Then

$$\Phi(x)(y) = \varphi_x(y) = \Psi(x, y) = \Psi(y, x) = \varphi_y(x) = \Phi(y)(x),$$

as required.

We claim that the homomorphism  $\Phi$  is one-to-one. Indeed, let  $g_1$  and  $g_2$  be distinct elements of  $H$ . Take a countable infinite set  $Y \subseteq X$  such that  $\{g_1, g_2\} \subseteq \langle Y \rangle$ , and let  $C = \langle Y \rangle$ . There exists a homomorphism  $f : C \rightarrow \mathbb{Z}(2)$  such that  $f(g_1 \cdot g_2) = -1$ . Since  $C$  is countable, Claim 1 implies that there exists  $x \in X$  such that the restriction of  $\varphi_x$  to  $C$  and  $f$  coincide. We now have that

$$-1 = f(g_1 \cdot g_2) = \varphi_x(g_1 \cdot g_2) = \varphi_x(g_1) \cdot \varphi_x(g_2),$$

whence it follows that  $\varphi_x(g_1) \neq \varphi_x(g_2)$ . Since the function  $\Psi$  is symmetric, we conclude that

$$\begin{aligned} \varphi_{g_1}(x) &= \Psi(g_1, x) = \Psi(x, g_1) = \varphi_x(g_1) \neq \varphi_x(g_2) \\ &= \Psi(x, g_2) = \Psi(g_2, x) = \varphi_{g_2}(x), \end{aligned}$$

that is,  $\varphi_{g_1}(x) \neq \varphi_{g_2}(x)$ . Hence  $\Phi(g_1) \neq \Phi(g_2)$  and  $\Phi$  is a monomorphism.

Let  $\tau$  be the topology on  $H$  generated by the family  $\{\varphi_g : g \in H\}$  of homomorphisms of  $H$  to  $\mathbb{Z}(2)$ . This is equivalent to saying that we identify  $H$  with the topological subgroup  $\varphi(H)$  of the compact group  $\mathbb{Z}(2)^H$ , where  $\varphi$  is the diagonal product of the family  $\{\varphi_g : g \in H\}$ . The group  $(H, \tau)$  is denoted by  $G$ . Claim 1 and Lemma 3.1 together imply that  $G$  is pseudocompact and each character  $\chi \in G^\wedge$  has the form  $\chi = \varphi_g$ , for some  $g \in G$ . This means that  $\Phi(G) = G^\wedge$  and, therefore,  $\Phi : G \rightarrow G^\wedge$  is an (abstract) isomorphism.

To conclude that  $\Phi$  is a topological isomorphism, it suffices to verify that  $\Phi$  is continuous and open. By Claim 1, all countable subgroups of  $G$  are  $h$ -embedded. Hence Proposition 2.1 of [1] implies that all compact subsets of  $G$  are finite. It now follows that the compact-open topology of the group  $G^\wedge$  and the topology of pointwise convergence on elements of  $G$  coincide. Equivalently, this means that the canonical base of  $G^\wedge$  at the neutral element consists of the sets

$$V(g_1, \dots, g_n) = \{\chi \in G^\wedge : \chi(g_k) = 1 \text{ for each } k = 1, \dots, n\},$$

where  $g_1, \dots, g_n \in G$ . Similarly, our definition of the topology on  $G$  implies that the sets

$$U(g_1, \dots, g_n) = \bigcap_{k=1}^n \varphi_{g_k}^{-1}(1),$$

with  $g_1, \dots, g_n \in G$ , constitute a local base at the neutral element of  $G$ . It follows from the definition of  $\Phi$  that

$$\begin{aligned} \Phi^{-1}(V(g_1, \dots, g_n)) &= \{h \in G : \Phi(h)(g_k) = 1 \text{ for } k = 1, \dots, n\} \\ &= \{h \in G : \varphi_h(g_k) = 1 \text{ for } k = 1, \dots, n\} \\ &= \{h \in G : \varphi_{g_k}(h) = 1 \text{ for } k = 1, \dots, n\} \\ &= \bigcap_{k=1}^n \varphi_{g_k}^{-1}(1) = U(g_1, \dots, g_n). \end{aligned}$$

We have shown, therefore, that  $\Phi^{-1}(V(g_1, \dots, g_n)) = U(g_1, \dots, g_n)$  for all  $g_1, \dots, g_n \in G$ . In addition, since  $\Phi$  is a mapping onto  $G^\wedge$ , it follows that  $\Phi(U(g_1, \dots, g_n)) = V(g_1, \dots, g_n)$ . These equalities and the fact that  $\Phi$  is an isomorphism imply that  $\Phi$  is a continuous open mapping of  $G$  onto  $G^\wedge$  or, equivalently,  $\Phi$  is a topological isomorphism. Since  $\Phi$  is symmetric, the group  $G$  is strongly self-dual.

Finally, our construction of  $G$  guarantees that  $|G| = |H| = |X| = \kappa$ , while the equality  $w(G) = \kappa$  follows from Proposition 2.7.  $\square$

**Remark 3.4.** We mentioned in the introduction that the group  $G$  in Theorem 3.3 is always zero-dimensional, independently of its size. Taking, for a given cardinal  $\kappa$  with  $\kappa^\omega = \kappa \geq \omega$ , a free Abelian group  $H$  of size  $\kappa$  and constructing a special symmetric  $\kappa \times \kappa$  matrix with entries in the circle group  $\mathbb{T}$ , one can topologize  $H$  in such a way that  $H$  will become a connected, pseudocompact self-dual group. Therefore, the self-duality of a group determines neither dimensional nor algebraic properties of the group.

**Remark 3.5.** The mere existence of arbitrarily big strongly self-dual pseudocompact groups can alternatively be established as follows. By Corollary 2.3 and Proposition 2.6 of [1], every pseudocompact Abelian group whose countable subgroups are  $h$ -embedded is reflexive and its dual group  $G^\wedge$  is pseudocompact. Now take the direct sum  $G \oplus G^\wedge$  which is pseudocompact by a Comfort–Ross theorem in [7]. Then  $H^\wedge \cong G^\wedge \oplus G^{\wedge\wedge} \cong G^\wedge \oplus G \cong H$ , so  $H$  is self-dual. In fact, the natural topological isomorphism  $\Phi: H \rightarrow H^\wedge$  is symmetric, so that  $H$  is strongly self-dual. It seems, however, that our construction in Theorem 3.3 is potentially more flexible.

#### 4. Problem section

Pseudocompactness implies several non-trivial restriction on the cardinality and algebraic structure of a topological Abelian group [9]. We do not know, apart from Proposition 2.7, whether self-duality adds something new to the known restrictions in the class of pseudocompact groups:

**Problem 4.1.** Suppose that an abstract Abelian group  $G$  admits a pseudocompact Hausdorff topological group topology. Does  $G$  admit a (strongly) self-dual pseudocompact Hausdorff topological group topology?

Every abstract Abelian group admits a precompact Hausdorff topological group topology [2, Theorem 1.4.25]. This fact motivates the following problem which contains four subproblems:

**Problem 4.2.** Characterize the algebraic structure of uncountable Abelian groups that admit a (strongly) self-dual precompact or pseudocompact Hausdorff topological group topology.

The restriction on the cardinality of groups in Problem 4.2 appears due to Corollary 2.8 which states that every infinite precompact self-dual group is uncountable.

We proved in Lemma 2.3 that all compact subsets of a pseudocompact self-dual group are finite. According to Remark 2.4, this assertion remains valid for precompact self-dual groups as well. We do not know, however, whether infinite self-dual precompact groups can be realcompact or have countable tightness:

**Problem 4.3.** Does there exist an infinite precompact self-dual group which is realcompact?

**Problem 4.4.** Can an infinite precompact self-dual group have countable tightness?

The last problem suggests a way to generalize Proposition 2.2:

**Problem 4.5.** Is every self-dual group  $G$  reflexive? What if  $G$  is additionally precompact or pseudocompact?

#### Acknowledgement

The author is grateful to Elena Martín-Peinador for helpful discussions regarding the results of this article.

#### References

- [1] S. Ardanza-Trevijano, M.J. Chasco, X. Domínguez, M.G. Tkachenko, Precompact non-compact reflexive Abelian groups, submitted for publication.
- [2] A.V. Arhangel'skii, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Studies in Mathematics, vol. I, Atlantis Press/World Scientific, Paris/Amsterdam, 2008.
- [3] W. Banaszczyk, Additive Subgroups of Topological Vector Spaces, Lecture Notes in Math., vol. 1466, Springer-Verlag, Berlin/Heidelberg, 1991.
- [4] M.J. Chasco, Pontryagin duality for metrizable groups, Arch. Math. 70 (1998) 22–28.
- [5] M.J. Chasco, E. Martín-Peinador, Pontryagin duality and strong duality, Extracta Math. 6 (1) (1991) 44–46.
- [6] M.J. Chasco, E. Martín-Peinador, On strongly reflexive topological groups, Appl. Gen. Topol. 2 (2001) 219–226.
- [7] W.W. Comfort, K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pacific J. Math. 16 (3) (1966) 483–496.
- [8] L. Corwin, Some remarks on self-dual locally compact Abelian groups, Trans. Amer. Math. Soc. 148 (2) (1970) 613–622.
- [9] D.N. Dikranjan, D.B. Shakhmatov, Algebraic structure of pseudocompact groups, Mem. Amer. Math. Soc. 133 (633) (1998) 1–83.
- [10] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
- [11] S. Hernández, Pontryagin duality for topological abelian groups, Math. Z. 238 (3) (2001) 493–503.
- [12] S. Hernández, S. Macario, Dual properties in totally bounded Abelian groups, Arch. Math. (Basel) 80 (2003) 271–283.
- [13] S. Hernández, V. Uspenskij, Pontryagin duality for spaces of continuous functions, J. Math. Anal. Appl. 242 (2000) 135–144.
- [14] E. Hewitt, K. Ross, Abstract Harmonic Analysis, vol. I, Springer-Verlag, Berlin/Göttingen/Heidelberg, 1979.
- [15] S. Kaplan, Extensions of the Pontryagin duality I: Infinite products, Duke Math. J. 15 (1948) 649–658.
- [16] H. Leptin, Abelsche Gruppen mit kompakten Charaktergruppen und Dualitätstheorie gewisser linear topologischer abelscher Gruppen, Abh. Math. Sem. Univ. Hamburg 19 (1955) 244–263.
- [17] V.G. Pestov, Free topological abelian groups and the Pontryagin duality, Moscow Univ. Math. Bull. 41 (1986) 1–4; Russian original in: Vestnik Mosk. Univ. Ser. I (1) 3–5.
- [18] V.G. Pestov, Free abelian topological groups and the Pontryagin–van Kampen duality, Bull. Austral. Math. Soc. 52 (1995) 297–311.
- [19] S.U. Raczkowski, F.J. Trigos-Arrieta, Duality of totally bounded Abelian groups, Bol. Soc. Mat. Mexicana (3) 7 (1) (2001) 1–12.
- [20] R. Venkataraman, Extensions of Pontryagin duality, Math. Z. 143 (2) (1975) 105–112.